Probability Theory
2014/15 Semester IIb
Instructor: Daniel Valesin
Reexamination
7/7/2015
Duration: 3 hours

Name: ______________________
Student number: __________

This exam contains 9 pages (including this cover page) and 7 problems. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You are allowed to have two hand-written sheets of paper and a calculator.

You are required to show your work on each problem except for Problem 1 (True or False).

Do not write on the table below.

Problem	Points	Score
1	12	
2	18	
3	12	
4	12	
5	12	
6	12	
7	12	
Total:	90	

Standard Normal cumulative distribution function
The value given in the table is $F_X(x)$ for $X \sim \mathcal{N}(0,1)$.

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

1. (12 points) In each of the items below, write (T) if the statement is true and (F) if it is false. No justification is required.

a) ( ) For any random variable X and real number x, $\mathbb{P}(X = x) = 0$ if and only if F_X is continuous at x.

b) ( ) A random variable is a subset of the sample space.

c) ( ) If X is a continuous random variable, then $\mathbb{P}(X \in \mathbb{N}) = 0$ ($\mathbb{N}$ denotes the set of natural numbers).

d) ( ) Let $(p_n)_{n\geq 1}$ be a sequence with $0 \leq p_n \leq 1$ for each n and

$$np_n \xrightarrow{n\to\infty} \lambda > 0.$$

Assume $Y \sim \text{Poisson}(\lambda)$ and, for each n, $X_n \sim \text{Binomial}(n, p_n)$. Then, for each k, we have

$$\mathbb{P}(X_n > k) \xrightarrow{n\to\infty} \mathbb{P}(Y > k).$$

e) ( ) If X_1 and X_2 are independent random variables with $X_1 \sim \text{Binomial}(n_1, p)$ and $X_2 \sim \text{Binomial}(n_2, p)$, then $X_1 + X_2 \sim \text{Binomial}(n_1 + n_2, p)$.

f) ( ) If X_1 and X_2 are independent random variables with $X_1 \sim \text{Poisson}(\lambda_1)$ and $X_2 \sim \text{Poisson}(\lambda_2)$, then $X_1 + X_2 \sim \text{Poisson}(\lambda_1 + \lambda_2)$.

g) ( ) If U is uniformly distributed on the interval $(-1, 1)$, then U^2 is uniformly distributed on the interval $(0, 1)$.

h) ( ) If X_1, X_2, X_3, X_4 are independent and $Y_1 = X_1^2 + X_2^3$, $Y_2 = X_3 - X_4$, then Y_1 and Y_2 are independent.

i) ( ) For any positive random variable X, we have $\mathbb{E}(\ln(X)) \geq \ln(\mathbb{E}(X))$ (provided the expectations exist).

j) ( ) Let X and Y be discrete random variables. Suppose we know the probability mass functions f_X and f_Y but do not know the joint probability mass function $f_{X,Y}$. Then, it is always impossible to tell whether or not X and Y are independent.

k) ( ) I toss a fair coin twice and write down each result ('heads' or 'tails') in separate pieces of paper. You take one of the two pieces of paper at random and it says 'heads'. The probability that the other piece of paper also says 'heads' is $\frac{1}{2}$.

l) ( ) If $X_1, X_2, \ldots$ are independent random variables and $\text{Var}(X_n) \leq 1$ for each n, then

$$\text{Var}\left(\frac{\sum_{i=1}^{n} X_i}{n}\right) \xrightarrow{n\to\infty} 0.$$

2. (a) (6 points) We want to distribute m candies to n children. Assume that the candies are indistinguishable and the children are distinguishable. Moreover, assume $m \geq n$ and each child must receive at least one candy. Determine the number of ways in which the distribution can occur.

   (b) (6 points) Given positive integers m and n, with $m \leq n$, determine the number of injective functions $f : \{1, \ldots, m\} \to \{1, \ldots, n\}$. *(A function is called injective if $f(x) \neq f(y)$ whenever $x \neq y$).*

   (c) (6 points) I have two coins: one is fair (that is, it gives heads with probability $\frac{1}{2}$) and the other gives heads with probability $\frac{1}{3}$. Assume you take one of the two coins at random and toss it until the first heads appears. If four tosses were required, what is the probability that you took the fair coin?

3. (a) (6 points) Assume X and Y are uniformly distributed on the set T, defined as the triangle in the xy-plane with vertices $(0,0)$, $(1,0)$ and $(1,2)$. That, is the joint density function of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} 1 & \text{if } (x,y) \in T; \\ 0 & \text{otherwise.} \end{cases}.$$

Find f_X, f_Y, $\mathbb{E}(\max(X,Y))$.

(b) (6 points) Assume U and V are independent random variables, both uniformly distributed on the interval $(0,1)$. Let $W = U \cdot V$. Find f_W.

4. (a) (6 points) Let $X \sim \text{Poisson}(\lambda)$ and Y have distribution given by

$$\mathbb{P}(Y = 0 \mid X = 0) = 1;$$

$$\mathbb{P}(Y = y \mid X = x) = \binom{x}{y} p^y (1-p)^{x-y}, \qquad x \in \{1, 2, \ldots\},\ y \in \{0, \ldots, x\}.$$

(We abbreviate this by writing: $Y|X = x \sim \text{Binomial}(x, p)$). Compute the moment generating function of Y and use it to prove that $Y \sim \text{Poisson}(\lambda p)$. You are allowed to use directly the formulas for the moment generating functions of the Binomial and Poisson distributions.

(b) (6 points) Let $X_1, \ldots, X_n$ be independent random variables, all following the exponential distribution with parameter 1. Find the density function of $Y = \max(X_1, \ldots, X_n)$.

5. In a field there are d ducks flying over h hunters. Each hunter chooses a duck uniformly at random and shoots, hitting (and killing) it with probability p. Assume that all the choices and shots of the hunters are independent (in particular, it is possible that several hunters choose and/or hit the same duck). Let X be the number of ducks that survive.
   (a) (7 points) Find $\mathbb{E}(X)$.
   (b) (5 points) Find $\mathrm{Var}(X)$.

6. I have a coin that gives heads with unknown probability p. I toss it n times (n large); for $1 \leq i \leq n$, let X_i be equal to 1 if the ith toss gave heads and 0 otherwise. Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

   (a) (6 points) Using a normal approximation, estimate:

   $$\mathbb{P}\left(|\bar{X}_n - p| > \frac{1}{5}\sqrt{\frac{p(1-p)}{n}}\right).$$

   (b) (6 points) For $m \leq n$, find the value of p that maximizes the probability of obtaining m heads out of n tosses. *(Hint. If $g(p)$ is a non-negative function, p maximizes $g(p)$ if and only if it maximizes* $\ln(g(p))$*.)*

7. (a) (6 points) State and prove the Weak Law of Large Numbers.

   (b) (6 points) Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with mean μ and variance σ^2. Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$. Show that

$$\mathbb{E}\left(\frac{1}{n-1}\sum_{i=1}^{n} X_i^2 - \frac{n}{n-1}\bar{X}_n^2\right) = \sigma^2.$$

   Hint. Recall that $\mathbb{E}(X_i^2) = \mathrm{Var}(X_i) + \mathbb{E}(X_i)^2 = \sigma^2 + \mu^2$.